MATH4210 Tutorial 7

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Outline

Normal r.v.

For a normal r.v. X with parameters μ, σ^2, the density function of X is given by

$$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}.$$

(a).Find $\mathbb{E}[X]$, $Var(X)$. (μ, σ^2)

(b).Find $\mathbb{E}[|X|]$, $\mathbb{E}[(X-K)^+]$.

(c).Find $\underline{\mathbb{E}[e^{\theta X}]}$, $\mathbb{E}[e^X]$.

(d).Find $\mathbb{E}[X^2]$.

$\frac{x-\mu}{\sigma} \Rightarrow x = \sigma y + \mu$

$dy = \frac{dx}{\sigma}$

(a) $E[X] = \mu$, $Var(X) = \sigma^2$.

(b) $E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$.

$$E[|X|] = \underbrace{\int_0^{\infty} x \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx}_{x \geq 0,\ I_1} + \underbrace{\int_{-\infty}^{0} (-x) \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx}_{x<0,\ I_2}$$

$y = \frac{x-\mu}{\sigma}$

Normal r.v.

$$I_1 = \int_{-\frac{\mu}{\sigma}}^{\infty} (\sigma y+\mu) \cdot \underbrace{\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}}_{\text{pdf of } N(0,1)} dy = \sigma \underbrace{\int_{-\frac{\mu}{\sigma}}^{\infty} y \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} dy}_{} + \mu \cdot \underbrace{\int_{-\frac{\mu}{\sigma}}^{\infty} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} dy}_{\mu \cdot (1-\Phi(-\frac{\mu}{\sigma}))}$$

$$= \sigma \cdot \int_{-\frac{\mu}{\sigma}}^{\infty} \left(-\frac{1}{\sqrt{2\pi}}\right) e^{-\frac{y^2}{2}} d\left(-\frac{y^2}{2}\right)$$

Φ: cdf of $N(0,1)$

$$= \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{\mu^2}{2\sigma^2}} + \mu \cdot \left(1-\Phi\left(-\frac{\mu}{\sigma}\right)\right)$$

$$I_2 = \frac{\sigma}{\sqrt{2\pi}} e^{-\frac{\mu^2}{2\sigma^2}} - \mu \cdot \Phi\left(-\frac{\mu}{\sigma}\right)$$

$$E[|X|] = \frac{2\sigma}{\sqrt{2\pi}} e^{-\frac{\mu^2}{2\sigma^2}} + \mu \cdot \left(1-2\Phi\left(-\frac{\mu}{\sigma}\right)\right).$$

$$E[(X-k)^+] = \int^{\infty} (x-k) f_X(x) dx$$

$$y^+ = \begin{cases} y, & y \ge 0 \\ 0 & \text{o.w.} \end{cases}$$

Assume $\underline{X \sim N(0,1)}$. $E[(X-K)^+] = \int_K^\infty (x-K)\cdot\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$

$$= \underbrace{\int_K^\infty x\cdot\frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}\,dx} - K\cdot\int_K^\infty \underbrace{\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}}_{\text{pdf of } N(0,1)}\,dx$$

$$= \int_K^\infty -\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,d\left(-\frac{x^2}{2}\right) \qquad K(1-\Phi(K)).$$

$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{K^2}{2}}.$$

$$= \frac{1}{\sqrt{2\pi}}e^{-\frac{K^2}{2}} - K(1-\Phi(K)).$$

For $X \sim N(\mu,\sigma^2)$. $X = \mu + \sigma Y$. $Y \sim N(0,1)$.

$$E[(X-K)^+] = E[(\sigma Y + \mu - K)^+] = \sigma\cdot \underbrace{E\left[\left(Y - \frac{K-\mu}{\sigma}\right)^+\right]}$$

$$= \sigma\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{(K-\mu)^2}{2\sigma^2}} - \frac{K-\mu}{\sigma}\left(1-\Phi\left(\frac{K-\mu}{\sigma}\right)\right)\right)$$

(3) $E[e^{\theta X}] = \int_{-\infty}^\infty e^{\theta x}\cdot\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2}}\,dx$

$$\overset{y=\frac{x-\mu}{\sigma}}{\underset{x=\sigma y+\mu}{=}} \int_{-\infty}^\infty e^{\theta(\sigma y+\mu)}\cdot \underbrace{\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}}_{\text{pdf of } N(0,1)}\,dy$$

$$= e^{\theta\mu}\cdot \int^\infty \frac{1}{\ } e^{-\frac{y^2}{2}+\theta\sigma y}\,dy$$

$$\int_{-\infty} \frac{1}{\sqrt{2\pi}} e \qquad dy.$$

Normal r.v.

$$-\frac{y^2}{2}+\theta\sigma y \qquad = e^{\theta\mu+\frac{\theta^2\sigma^2}{2}} \cdot \underbrace{\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{(y-\theta\sigma)^2}{2}} dy}_{\text{pdf of } N(\theta\sigma,1).} = 1$$

$$= -\frac{1}{2}(y-\theta\sigma)^2+\frac{\theta^2\sigma^2}{2}$$

$$= e^{\theta\mu+\frac{\theta^2\sigma^2}{2}}$$

$$E[e^X] = e^{\mu+\frac{\sigma^2}{2}}$$

(4) $E[X^2] = \underbrace{\int_{-\infty}^{\infty} x^2 \cdot \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx}$. (Left as exercise).

$$E[X^2] = Var(X)+(E[X])^2 = \sigma^2+\mu^2.$$

Normal r.v.

Suppose $X_k \sim N(\mu_k,\ \sigma_k^2)$ and $\mathbb{E}[|X_k - X|^2] \to 0$ as $K \to \infty$, then X is a normal random variable with $\mathbb{E}[X] = \lim \mu_k$ and $Var(X) = \lim \sigma_k^2$.

Proof:
From $\mathbb{E}[|X_k - X|^2] \to 0$, we have $\mathbb{E}[X_k] \to \mathbb{E}[X]$ and $Var(X_k) \to Var(X)$.
Since $|e^{iax} - e^{iay}| \leq |a| \cdot |x - y|$, we have

$$\mathbb{E}\big[\{e^{i\theta X_k} - e^{i\theta X}\}^2\big] \leq |\theta|^2 \cdot \mathbb{E}[|X_k - X|^2] \to 0 \ \text{ as } n \to \infty.$$

Therefore

$$\mathbb{E}[e^{i\theta X_k}] \to \mathbb{E}[e^{i\theta X}] \ \ k \to \infty.$$

So X is normal, with mean $\mathbb{E}[X] = \lim \mu_k$ and $Var(X) = \lim \sigma_k^2$.

Proof: Since $E[|X_k-X|^2] \to 0$.

By Cauchy-Schwarz inequality:

$$\left|E[X_k]-E[X]\right| \le \sqrt{E[|X_k-X|^2]} \to 0. \Rightarrow E[X_k] \to E[X].$$

$$\sqrt{E[X^2]} - \sqrt{E[|X_k-X|^2]} \le \sqrt{E[X^2]} \le \sqrt{E[|X_k-X|^2]} + \sqrt{E[X_k^2]}.$$ (Minkowski's inequality)

$Var(X_k) \to Var(X)$.

By $\left|e^{iax}-e^{iay}\right| \le |a|\cdot|x-y|$.

$$\left|e^{i\theta X_k}-e^{i\theta X}\right|^2 \le \theta^2\cdot|X_k-X|^2.$$

Take expectation on both sides.

$$E[|e^{i\theta X_k}-e^{i\theta X}|^2] \le \theta^2\cdot E[|X_k-X|^2] \to 0.$$

$$E[e^{i\theta X_k}] \to E[e^{i\theta X}].$$

So X is a normal r.v., $E[X]=\lim \mu_k$, $Var(X)=\lim \sigma_k^2$.

Convergence

Let $(\Omega, \mathcal{F}, P)$ be a probability space and X_n and X are random variables from $(\Omega, \mathcal{F}, P)$ to $\mathbb{R}$. There are different notation of convergence:

Convergence almost surely

$\{X_n\}$ converges to X a.s. if

$$P(\omega \in \Omega : \lim_{n\to\infty} X_n(\omega) = X(\omega)) = 1.$$

Denoted as $X_n \xrightarrow{a.s.} X$.

Convergence in Probability

if for every $\rho > 0$,

$$\lim_{n\to\infty} P(\{\omega\in\Omega : |X_n(\omega)-X(\omega)| \geq \rho\}) = 0.$$

$$\lim_{n\to\infty} P(|X_n - X| \geq \rho\}) = 0.$$

Denoted as $X_n \xrightarrow{P} X$

Convergence

Convergence in Lp-norm

Given a real number $p \geq 1$, $\{X_n\}$ is said to converge to X in L^p-norm, if

$$\lim_{n\to\infty} \mathbb{E}[|X_n - X|^p] = 0.$$

Denoted as $X_n \xrightarrow{L^p} X$

Convergence in law (in distribution)

Let F_n and F denote the cumulative distribution function of X_n and X. $\{X_n\}$ is said to converge to X in law if

$$\lim_{n\to\infty} F_n(x) = F(x)$$

for every $x \in \mathbb{R}$ at which F is continuous.
Denoted as $X_n \xrightarrow{\mathcal{L}} X$.

(a) If $\{X_n\}$ converges to X a.s., then $\{X_n\}$ converges to X in probability.(Use Egorov's theorem)

(b)If $\{X_n\}$ converges to X in probability, then $\{X_n\}$ converges to X in distribution.

Egorov thm: Let $(X, \mathcal{M}, \mu)$ be a measure space with $\mu(X) < \infty$. Let $\{f_n\}$ be a sequence of measurable functions on X and let f be a measurable function on X. Assume that $f_n \to f$ a.e. pointwise. Then for any $\epsilon > 0$, there exists a measurable set D of X, such that $\mu(D) < \epsilon$ and $f_n \to f$ uniformly on $X - D$.

(a) $X_n \xrightarrow{a.s.} X \Rightarrow X_n \to X$ in probability.

Proof: By Egorov's thm,

$\forall \varepsilon > 0, \exists D \subseteq \Omega, P(D) < \varepsilon.$ $X_n \to X$ uniformly on Ω / D.

For any $\rho > 0$, $\exists N$, s.t. $|X_n - X| < \rho$ on Ω / D.

$P(w : |X_n(w) - X(w)| \geq \rho) \leq P(D) < \varepsilon.$

$$\Rightarrow \lim_{n\to\infty} P(\omega: |X_n(\omega)-X(\omega)| \geq \varepsilon) = 0.$$

(b). $X_n \to X$ in probability, $X_n \xrightarrow{\mathcal{L}} X$.

$$F_n(x) = P(X_n \leq x) = P(X_n \leq x, X \leq x+\varepsilon) + P(X_n \leq x, X > \varepsilon)$$

$$\leq P(X \leq x+\varepsilon) + \underbrace{P(|X_n - X| > \varepsilon)}_{X_n \to X \text{ in prob.}} \quad (1)$$

$$F(x-\varepsilon) = P(X \leq x-\varepsilon) \leq \underbrace{P(X_n \leq x)}_{F_n(x)} + P(|X_n - X| > \varepsilon) \quad (2)$$

$$F(x-\varepsilon) \leq \liminf_{n\to\infty} F_n(x) \leq \limsup_{n\to\infty} F_n(x) \leq P(X \leq x+\varepsilon).$$

Let $\varepsilon \downarrow 0$, since F is continuous at x.

$$\lim_{n\to\infty} F_n(x) = F(x).$$

(c) If $\{X_n\}$ converges to X in L^p, then $\{X_n\}$ converge to X in probability.

(d)If $\{X_n\}$ converges to X in $L^2(X)$, then $\{X_n\}$ converges to X in $L^1(X)$.

(e) If $\{X_n\}$ converges to X in probability, then there exists a subsequence $\{X_{n_k}\}$ such that $\{X_{n_k}\}$ converges to X a.s. .(Google Borel-Cantelli lemma, left as exercise).

Proof:

(c). $X_n \xrightarrow{L^p} X \Rightarrow X_n \to X$ in probability.

$$P(w: |X_n(w) - X(w)| \geq \rho) = P(w: |X_n(w) - X(w)|^p \geq \rho^p)$$

$$\leq \frac{E(|X_n - X|^p)}{\rho^p} \to 0.$$

By Markovian inequality

(d). $X_n \xrightarrow{L^2} X \Rightarrow X_n \xrightarrow{L^1} X$.

By Cauchy-Schwarz inequality,

$$E[|X_n - X|] \leq \sqrt{E[|X_n - X|^2]}.$$

Log-normal r.v.

A random variable S is log-normal if $lnS \sim N(\mu, \sigma^2)$
(a) The probability density function $p_S(x)$ of S is given by

$$p_S(x) = \frac{1}{x\sqrt{2\pi}\sigma} \cdot e^{-\frac{(lnx-\mu)^2}{2\sigma^2}}$$

Solution: $P(S \le x)$.

If $x \le 0$. $P(S \le x) = 0$.

If $x > 0$. $P(S \le x) = P(\ln S \le \ln x)$

$$= \int_{-\infty}^{\ln x} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt$$

$$\overset{z=\frac{y-\mu}{\sigma}}{=\!=\!=} \int_{-\infty}^{\frac{\ln x-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz$$

$$P_S(x) = \frac{dP(S\le x)}{dx} = \frac{d\int_{-\infty}^{\frac{\ln x-\mu}{\sigma}} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz}{dx}$$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{(\ln x-\mu)^2}{2\cdot\sigma^2}} \cdot \frac{d\frac{\ln x-\mu}{\sigma}}{dx}$$

$$= \frac{1}{\sqrt{2\pi}\cdot x\sigma} e^{-\frac{(\ln x-\mu)^2}{2\sigma^2}}.$$